Journal of Statistical Physics, Vol. 46, Nos. 3/4, 1987

On the Existence of Thermodynamics for the Generalized Random Energy Model

D. Capocaccia,[1,2] M. Cassandro,[1,3] and P. Picco[1]

Received January 14, 1986; revision received June 4, 1986

Derrida's generalized random energy model is considered. Almost sure and L^p convergence of the free energy at any inverse temperature β are proven for an arbitrary number n of hierarchical levels. The explicit form of the free energy is given in the most general case and the limit $n \to \infty$ is discussed.

KEY WORDS: Random energy; random variables; spin glass.

1. INTRODUCTION

Random energy models were introduced by Derrida[1] as simple and solvable models for spin glasses. Spin glasses are disordered magnetic systems and, as is well known, a mean field description of such system is given by the Sherrington–Kirkpatrick model.[4] This model is defined on a one-dimensional lattice for Ising spin $\sigma = \pm 1$ by the Hamiltonian

$$H(\boldsymbol{\sigma}) = -\sum_{(i,j)} J_{ij} \sigma_i \sigma_j$$

where the J_{ij} are independent Gaussian random variables with zero mean and variance $1/\sqrt{N}$. The quantity $H(\boldsymbol{\sigma})$, the energy associated with a given configuration $\boldsymbol{\sigma}$, is a Gaussian random variable with

$$\mathbb{E}(H) = 0, \qquad \mathbb{E}([H(\boldsymbol{\sigma})]^2) = N, \qquad \mathbb{E}(H(\boldsymbol{\sigma})\, H(\boldsymbol{\sigma}')) = \frac{1}{N}\left(\sum_{i=1}^{N} \sigma_i \sigma_i'\right)^2$$

[1] Centre de Physique Théorique, Centre National de la Recherche Scientifique, Luminy, Marseille, France.

[2] Dipartimento di Matematica, Università di Roma "La Sapienza," Rome, Italy, and CNR, GNFM.

[3] Dipartimento de Fisica, Università di Roma "La Sapienza," Rome, Italy, and CNR, GNSM, Université de Provence.

No rigorous results exist for this model, but the random energy model can be rigorously solved.[5]

The GREM[2] is the most recent model and it describe a system with 2^N possible configurations where the Hamiltonian is not specified, but the energy associated with each configuration is the sum of Gaussian random variables and the correlations between different configurations are described by a covariance function that depends on $n-1$ parameters.

With a suitable choice of the above-mentioned parameters, it is possible to reproduce, in the limit $n \to \infty$, many features that are typical of "Hamiltonian" spin-glass models.[3,4] In Ref. 2 a method for evaluating the infinite-volume limit of the average free energy for arbitrary n is described and some examples of explicit solutions are exhibited.

In this paper we show that the sequence of random variables

$$F_N(\beta) = \frac{1}{N} \ln Z_N = \frac{1}{N} \ln \sum_{\nu=1}^{2^N} \exp \beta \varepsilon_\nu$$

converges almost surely to its average value $F(\beta)$ and in all $\mathbb{L}^p$, $1 \leqslant p < \infty$, and we give the explicit form of $F(\beta)$ in the most general case. We prove also that it is sufficient to have $n/N = o[(\ln N)^{1+\eta}]$ to get the same results when the joint limit $N \to \infty$ and $n \to \infty$ is considered.

In Section 2 we define the model and state our theorem. The proofs are given in Section 3.

The strategy is similar to that used in Ref. 5 for $n = 1$. In that case the main ingredient of the proof was that for large N the maximum of the sequence of independant Gaussian random variables $E_1/N,..., E_{2^N}/N$ is almost surely bounded. In our case each energy is the sum of n independent Gaussian random variables and we show that there exists a compact, convex region Q in $\mathbb{R}^n$ where typically the 2^N points associated with the sequence $E_1,..., E_{2^N}$ lie (cf. Proposition 3.1).

We also prove a strong law of large numbers for the occupation number of the neighborhood of an arbitrary point in Q (cf. Proposition 3.4). By the use of suitable lower and upper bounds we show that almost surely

$$\lim_{N \to \infty} F_N(\beta) = F(\beta) = \lim_{N \to \infty} \frac{1}{N} \ln[\mathbb{E}(Z_N \chi_Q^N)]$$

where $\mathbb{E}$ is the average with respect to the Gaussian random variables and χ_Q^N is the characteristic function of the set $E_\nu \in Q$, $\forall \nu \in \{1,..., 2^N\}$. The asymptotic behavior of $\mathbb{E}(Z_N \chi_Q^N)$ is obtained by very simple geometrical considerations. The $\mathbb{L}^p$ convergence follows from the mean convergence theorem by proving the uniform integrability of the random variables $|F_N(\beta)|^p$.

2. DEFINITIONS AND RESULTS

For any $n \in \mathbb{N}$ and any $N \in \mathbb{N}$ let $a_i \geqslant 0$ and $\alpha_i \geqslant 1$, $i = 1, ..., n$, be real, positive numbers such that

$$\sum_{i=1}^{n} a_i = 1, \qquad \sum_{i=1}^{n} \ln \alpha_i = \ln 2$$

and let $(\Omega, \Sigma, \mathbb{P})$ be a probability space such that for any $n \in \mathbb{N}$ and $N \in \mathbb{N}$ there exists a family of $\alpha_1^N + \alpha_1^N \alpha_2^N + \cdots + \alpha_1^N \cdots \alpha_n^N$ independent, normalized, Gaussian random variables

$$\varepsilon^j_{k_1,...,k_j} \in \mathcal{N}(0, 1), \qquad j = 1, ..., n, \quad k_j = 1, ..., \alpha_j^N$$

defined on $(\Omega, \Sigma, \mathbb{P})$.

The GREM at inverse temperature β is then defined by the family of random variables

$$Z_{n,N}(\beta) = \sum_{k_1=1}^{\alpha_1^N} \cdots \sum_{k_n=1}^{\alpha_n^N} \exp \beta N^{1/2} \left(\sum_{j=1}^{n} a_j^{1/2} \varepsilon^j_{k_1,...,k_j} \right) \tag{2.1}$$

Define also, for any $k \in \{1, ..., n\}$, the following subset of $\mathbb{R}^k$:

$$\tilde{\mathscr{E}}(k) = \left\{ (X_i), i = 1, ..., k \mid \forall j \in 1, ..., k, \sum_{i=1}^{j} X_i^2 \leqslant 2 \sum_{i=1}^{j} \ln \alpha_i \right\} \tag{2.2}$$

and $\|x\|^2 = \sum_{i=1}^{n} X_i^2$ is the Euclidean norm of $\mathbb{R}^n$. Our main result is:

Theorem 2.1. Let m^* be the n-dimensional vector, $m^* \equiv (m_i^*)_{i=1}^n = (\beta a_i^{1/2})_{i=1}^n$, and let m be the point of the compact, convex subset $\tilde{\mathscr{E}}(n)$ of $\mathbb{R}^n$ at minimal distance from m^*. Then for any $\beta > 0$

$$F_n \equiv \lim_{N \to \infty} \frac{1}{N} \ln Z_{n,N}(\beta) = \frac{\|m^*\|^2}{2} - \frac{\|m - m^*\|^2}{2} + \ln 2 \tag{2.3}$$

almost surely and in $\mathbb{L}^p(\Omega, \Sigma, \mathbb{P})$ for any $1 \leqslant p < \infty$. If we consider the joint limit $N \to \infty$ and $n \to \infty$, the same result holds if $n \leqslant \text{const} \times N/(\ln N)^{1+\eta'}$ with $\eta' > 0$.

3. PROOF OF THEOREM 2.1

The proof of Theorem 2.1 follows from Propositions 3.3 and 3.5 and inequalities (3.6) and (3.7). We study first the almost sure convergence.

Let us first consider the case where n is fixed. We start with an upper bound for the free energy.

Given a family $\boldsymbol{\delta} = (\delta_1,..., \delta_n)$ of strictly positive numbers and $j \in 1,..., n$, let

$$Q_j = \sum_{i=1}^{j} \ln \alpha_i + \delta_j$$

and

$$A(j, N) = \left\{ \omega \in \Omega \mid \forall k_1,..., k_j, \sum_{i=1}^{j} (\varepsilon^i_{k_1,...,k_i})^2 \leqslant 2Q_j N \right\}$$

Let us define $\chi(Q_j, N) \equiv \mathbb{1}_{A(j,N)}$ as the characteristic function of the set $A(j, N)$.

Proposition 3.1. For any $\boldsymbol{\delta} = (\delta_1,..., \delta_n) \in \mathbb{R}^n$, $\delta_i > 0$, $\forall i = 1,..., n$, there exists $\Omega_1 \subset \Omega$ such that $\mathbb{P}(\Omega_1) = 1$ and for any $\omega \in \Omega_1$, $\exists N_1(\omega, \boldsymbol{\delta})$ such that, for any $N \geqslant N_1(\omega, \delta)$,

$$\prod_{j=1}^{n} \chi(Q_j, N) = 1$$

Proof. Since

$$\mathbb{P}\left(\prod_{j=1}^{n} \chi(Q_j N) = 0\right) = \mathbb{P}\left(\bigcup_{j=1}^{n} A^c(j, n)\right) \leqslant \sum_{j=1}^{n} \mathbb{P}(A^c(j, n))$$

and

$$\mathbb{P}(A^c(j, n)) = \mathbb{P}\left(\operatorname{Max}_{k_1,...,k_j} \sum_{i=1}^{j} (\varepsilon^i_{k_1,...,k_i})^2 \geqslant 2Q_j N\right)$$
$$\leqslant e^{-\lambda Q_j N} \alpha_1^N \cdots \alpha_j^N (1-\lambda)^{-J/2}, \qquad \forall\, 0 < \lambda < 1$$

where we have used the exponential Chebyshev inequality and the fact that

$$\mathbb{E}\left(\exp \frac{\lambda}{2} \operatorname{Max}_{k_1,...,k_j} \sum_{i=1}^{j} (\varepsilon^i_{k_1,...,k_j})^2\right) \leqslant \alpha_1^N \cdots \alpha_j^N \mathbb{E}\left(\exp \frac{\lambda}{2} \sum_{i=1}^{j} (\varepsilon^i)^2\right)$$
$$= (\alpha_1^N \cdots \alpha_j^N)(1-\lambda)^{-J/2}$$

Choosing $\lambda = 1 - \eta$, with

$$\eta < \min_{1<j<n} \frac{\delta_j}{2(\delta_j + \sum_{i=1}^{j} 2 \ln \alpha_i)}$$

we get

$$\mathbb{P}(A^c(j, N)) \leqslant (\eta)^{-J/2} \exp(-\delta_j N/2), \qquad \forall j \in 1,..., n$$

Therefore

$$\sum_{N \geqslant 1} \sum_{j=1}^{n} \mathbb{P}(A^c(j, N)) < +\infty$$

and the result follows from the Borel–Cantelli lemma.

Proposition 3.2. The exists $\Omega_2 \subset \Omega$ such that $\mathbb{P}(\Omega_2) = 1$ and for any $\omega \in \Omega_2$ and any $\gamma > 0$, $\exists N_2(\omega, \gamma)$ such that if $N > N_2(\omega, \gamma)$, then

$$Z_{n,N}(\beta) \leqslant [\exp(\gamma N)] \left(\frac{1}{(2\pi)^{1/2}}\right)^n \int_{\substack{\varepsilon_1^2 \leqslant 2Q_1 N \\ \vdots \\ \varepsilon_1^2 + \cdots \varepsilon_n^2 \leqslant 2Q_n N}} d\varepsilon_1 \cdots d\varepsilon_n$$

$$\times \exp\left[\sum_{i=1}^{n} \left(N \ln \alpha_i - \frac{\varepsilon_i^2}{2}\right) + \beta \sum_{i=1}^{n} (Na_i)^{1/2} \varepsilon_i\right] \tag{3.1}$$

Proof. By the Markov inequality

$$\mathbb{P}\left\{\left(\prod_{j=1}^{n} \chi(Q_j, N)\, Z_{n,N}(\beta)\right) \geqslant e^{\gamma N} \mathbb{E}\left(\prod_{j=1}^{n} \chi(Q_j, N)\, Z_{n,N}(\beta)\right)\right\} \leqslant e^{-\gamma N}$$

The result follows from Proposition 3.1 and the Borel–Cantelli lemma.

Let us define

$$\tilde{\mathscr{E}}(k, \delta) = \left\{(X_i),\ i = 1,..., k \mid \forall j = 1,..., k,\ \sum_{i=1}^{j} X_i^2 \leqslant 2Q_j\right\}, \qquad k = 1,..., n$$

Proposition 3.3. Let $m^*(\beta)$ be as in Theorem 2.1 and let $m(\delta)$ be the point of the compact, convex subset $\tilde{\mathscr{E}}(k, \delta)$ of $\mathbb{R}^n$ at minimal distance from m^*. Then for any $\eta > 0$ and for any ω belonging to Ω_2 the following inequality holds:

$$\begin{aligned} \ln Z_{n,N}(\beta) &\leqslant N[F_n(\beta) + \eta + \gamma] \\ &\equiv N[\tfrac{1}{2}\|m^*\|^2 - \tfrac{1}{2}\|m - m^*\|^2 + \log \eta + \eta + \gamma] \end{aligned}$$

if $\delta = \sum_{i=1}^{n} \delta_i$ is small enough and $N \geqslant N_2(\omega)$, where γ is defined in Proposition 3.2.

Proof. Let us first remark that, since $\tilde{\mathscr{E}}(n, \delta)$ is convex, then, if $\varepsilon \equiv (\varepsilon_i)_{i=1}^n \in \tilde{\mathscr{E}}(n, \delta)$,

$$\sum_{i=1}^{n} (\varepsilon_i - \sqrt{N} m_i^*)^2 \geqslant \sum_{i=1}^{n} [\varepsilon_i - \sqrt{N} m_i(\delta)]^2 + \sum_{i=1}^{n} N[m_i(\delta) - m_i^*]^2 \tag{3.2}$$

Since (3.1) also can be written as

$$Z_{n,N}(\beta) \leqslant 2^N[\exp(\gamma N)] \left(\frac{1}{(2\pi)^{1/2}}\right)^n \int \chi_{\mathscr{E}(n,\delta)}\left(\frac{\varepsilon}{\sqrt{N}}\right)$$
$$\times \exp\left[\sum_{i=1}^{n} -\frac{1}{2}(\varepsilon_i - \sqrt{N}m_i^*)^2 + \frac{Nm_i^{*2}}{2}\right] d\varepsilon_1 \cdots d\varepsilon_n$$

then from inequality (3.2) one gets

$$Z_{n,N}(\beta) \leqslant 2^N \exp(\gamma N) \exp - \sum_{i=1}^{n} \frac{1}{2}[\sqrt{N}m_i(\delta) - \sqrt{N}m_i^*]^2 + N\sum_{i=1}^{n}\frac{m_i^{*2}}{2}$$

and the result follows from the continuity properties of $m_i(\boldsymbol{\delta})$ as a function of $\boldsymbol{\delta}$ together with Proposition 3.2.

To prove a lower bound for the free energy, we start with the following result:

Proposition 3.4. Let $\Delta = (\Delta_1, \ldots, \Delta_n)$ be a subset of $\mathbb{R}^n$ such that for any $j = 1, \ldots, n$

$$\prod_{i=1}^{j} \alpha_i^N \mathbb{E}(\mathbb{1}_{\Delta_i}(\varepsilon^i)) \geqslant \mathscr{I}_j(N)$$

where $\mathscr{I}_j(N)$ are such that $\sum_{N\geqslant 1}\sum_{j=1}^{n}[\mathscr{I}_j(N)]^{-1} < \infty$, then for any given $0 < \eta < 1$ there exist $\Omega_3 \subset \Omega$ with $\mathbb{P}(\Omega_3) = 1$ and for any $\omega \in \Omega_3$, $\exists N_3(\omega, \eta)$ such that $\forall N > N_3(\omega, \eta)$

$$\sum_{k_1=1}^{\alpha_1^N} \mathbb{1}_{\Delta_1}(\varepsilon_{k_1}^1) \sum_{k_2=1}^{\alpha_2^N} \mathbb{1}_{\Delta_2}(\varepsilon_{k_1k_2}^2) \cdots \sum_{k_n=1}^{\alpha_n^N} \mathbb{1}_{\Delta_n}(\varepsilon_{k_1-k_n}^n) \geqslant (1-\eta)\prod_{i=1}^{n}\alpha_i^N \mathbb{E}(\mathbb{1}_{\Delta_i}(\varepsilon^i))$$

Proof. Let us remark that

$$\mathbb{P}\left(\sum_{k_1}\mathbb{1}_{\Delta_1}\cdots\sum_{k_n}\mathbb{1}_{\Delta_n} \leqslant (1-\eta)\prod_{i=1}^{n}\alpha_i^N\mathbb{E}(\mathbb{1}_{\Delta_i})\right)$$
$$\leqslant \mathbb{P}\left(\left|\sum_{k_1}\mathbb{1}_{\Delta_1}\cdots\sum_{k_n}\mathbb{1}_{\Delta_n} - \mathbb{E}\left(\sum_{k_1}\mathbb{1}_{\Delta_1}\cdots\sum_{k_n}\mathbb{1}_{\Delta_n}\right)\right|\right.$$
$$\left.\geqslant \eta\prod_{i=1}^{n}\alpha_i^N\mathbb{E}(\mathbb{1}_{\Delta_i})\right) \tag{3.3}$$

and

$$D^2 = \mathbb{E}\left(\left[\sum_{k_1} \mathbb{1}_{\Delta_1} \cdots \sum_{k_n} \mathbb{1}_{\Delta_n} - \mathbb{E}\left(\sum_{k_1} \mathbb{1}_{\Delta_1} \cdots \sum_{k_n} \mathbb{1}_{\Delta_n}\right)\right]^2\right)$$

$$= \sum_{j=1}^{n-1} \sum_{k_1 \cdots k_j} \sum_{k_{j+1} \neq l_{j+1}} \sum_{\substack{k_{j+2} \cdots k_n \\ l_{j+2} \cdots l_n}} \mathbb{E}(\mathbb{1}_{\Delta_1}) \cdots \mathbb{E}(\mathbb{1}_{\Delta_j})$$

$$\times [1 - \mathbb{E}(\mathbb{1}_{\Delta_1}) \cdots \mathbb{E}(\mathbb{1}_{\Delta_j})][\mathbb{E}(\mathbb{1}_{\Delta_{j+1}}) \cdots \mathbb{E}(\mathbb{1}_{\Delta_n})]^2$$

$$\leqslant \left[\prod_{i=1}^{n} \alpha_i^N \mathbb{E}(\mathbb{1}_{\Delta_i})\right] \sum_{j=1}^{n-1} \alpha_{j+1}^N \mathbb{E}(\mathbb{1}_{\Delta_{j+1}}) \cdots \alpha_n^N \mathbb{E}(\mathbb{1}_{\Delta_n})$$

where we have used the independence of the ε's.

It follows from the Chebyshev inequality that the left-hand side of (3.3) does not exceed

$$\frac{1}{\eta^2} \sum_{j=1}^{n} \frac{1}{\prod_{i=1}^{j} \alpha_i^N \mathbb{E}(\mathbb{1}_{\Delta_i})} \leqslant \frac{1}{\eta^2} \sum_{j=1}^{n} [\mathscr{I}_j(N)]^{-1}$$

and the result follows from the Borel–Cantelli lemma.

Proposition 3.5. $\forall \rho > 0$, $\forall 1 > \eta > 0$ there exists $\Omega_4 \subset \Omega$ such that $\mathbb{P}(\Omega_4) = 1$ and $\forall \omega \in \Omega_4$, $\exists N_4(\omega, \eta)$ such that for any $N \geqslant N_4(\omega, \eta)$

$$Z_{n,N}(\beta) \geqslant (1 - \eta) \exp N[F(\beta) - \rho] \tag{3.4}$$

Proof. Let us remark first that $\forall \mathbf{\Delta} = (\Delta_1, ..., \Delta_n) \subset \mathbb{R}^n$

$$Z_{n,N}(\beta) \geqslant \sum_{k_1=1}^{\alpha_1^N} \mathbb{1}_{\Delta_1}(\varepsilon_{k_1}^1) \cdots \sum_{k_n=1}^{\alpha_n^N} \mathbb{1}_{\Delta_n}(\varepsilon_{k_1 - k_n}^n)$$

$$\times \exp \beta\sqrt{N} \operatorname*{Inf}_{\substack{\varepsilon^i \in \Delta_i \\ u = 1 - n}} \sum_{i=1}^{n} \sqrt{a_i}\, \varepsilon_i \tag{3.5}$$

Then choose

$$\mathbf{\Delta}(\boldsymbol{\rho}) = \{\boldsymbol{\varepsilon} \in \mathbb{R}^n \mid \sqrt{N}[m_i - 2\rho_i] < \varepsilon_i < \sqrt{N}[m_i - \rho_i], \forall i = 1 - n\}$$

where $(m_i)_{i=1}^n = m$ is defined in Theorem 2.1 and the $\{\rho_i\}_{i=1}^n$ are such that $\mathbf{\Delta}(\boldsymbol{\rho}) \subset \widetilde{\mathscr{E}}(n)$. Since

$$\mathbb{E}(\mathbb{1}_{\Delta_i}(\varepsilon^i)) \geqslant \frac{\exp - \frac{1}{2}N(m_i - 2\rho_i)^2}{2\sqrt{N}(m_i - \rho_i)(2\pi)^{1/2}}$$

we get

$$\prod_{i=1}^{j} \alpha_i^N \mathbb{E}(\mathbb{1}_{\Delta_i}) \leqslant \left(\frac{1}{2(2\pi N)^{1/2}}\right)^j \frac{\exp \sum_{i=1}^{j} 2N(\rho_i m_i - \rho_i^2)}{\prod^j (m_i - \rho_i)}$$

i.e., the hypotheses of Proposition 3.4 are satisfied. Then, using Eq. (3.5) and proposition (3.4), we get

$$Z_{n,N}(\beta) \geqslant (1-\eta) \exp N \left\{ \sum_{i=1}^{n} \left[\ln \alpha_i - \frac{1}{2}(m_i - 2\rho_i)^2 + \beta \sum_{i=1}^{n} a_i^{1/2}(m_i - 2\rho_i) \right] \right\}$$
$$= (1-\eta) \exp N \left[F_n(\beta) - \beta 2 \sum_{i=1}^{n} a_i^{1/2} \rho_i + 2 \sum_{i=1}^{n} \rho_i^2 + 2 \sum_{i=1}^{n} \rho_i m_i \right]$$

from which we get the result.

Remark. To study the joint limit $N \to \infty$ and $n \to \infty$, we have to show that in this limit all the previous probability estimates hold and that error estimates on the free energy can still be made as small as we please.

The error on the upper bound for the free energy is proportional to $\sum_{i=1}^{n} \delta_i$.

The error on the lower bound for the free energy is

$$\left| 2\beta \sum_{i=1}^{n} a_i^{1/2} \rho_i - 2 \sum_{i=1}^{n} \rho_i^2 - 2 \sum_{i=1}^{n} \rho_i m_i \right|$$

In order to prove the analogue of Proposition 3.1, it is sufficient to show that

$$\sum_{N \geqslant 1} \sum_{i=1}^{n} (\eta)^{-j} \exp(-\delta_j N/4) < \infty \tag{3.6}$$

To satisfy the hypotheses of Proposition 3.4, it is sufficient that

$$\sum_{N \geqslant 1} \sum_{j=1}^{n} (2\sqrt{N})^j \left\{ \prod_{i=1}^{j} [m_i(\beta) - \rho_i] \right\} \exp \left[- \sum_{i=1}^{j} 2N(\rho_i m_i - \rho_i^2) \right] < \infty \tag{3.7}$$

It is easy to check that if we choose

$$\delta_j = \frac{\delta}{j(\ln, j)^{1+\eta'}} \qquad \text{for some} \quad \eta' > 0$$

$$\rho_1 = \rho, \qquad \rho_i = \rho/\sqrt{N} \quad \text{for } i = 2,..., n, \qquad \eta = \delta_n/(4 \ln 2)$$

and we assume $n \leqslant \text{const} \times N/(\ln N)^{1+\eta''}$ for some $\eta'' > 0$, then the above-mentioned errors on the free energy can be made as small as we please for ρ and δ suitable and the two sums (3.6) and (3.7) are convergent.

3.1. Convergence in $\mathbb{L}^p(\Omega, \Sigma, \mathbb{P})$, $\forall p \in [1 \infty[$

In order to prove $\mathbb{L}^p$ convergence, we use the mean convergence criterion[6]: if the random variables $\{|y_N|^p, N \geqslant 1\}$ are uniformly integrable (u.i.), that is,

$$\lim_{\alpha \to \infty} \sup_{N \geqslant N_0} \int_{|y_N|^p \geqslant \alpha} |y_N|^p \, d\mathbb{P} = 0$$

and $y_N \to y$ in probability, then $u_N \to y$ in $\mathbb{L}^p$.

Since

$$F_N(\beta) \leqslant \beta N^{-1/2} \operatorname*{Max}_{k_1,\dots,k_n} \left(\sum_{j=1}^{n} a_j^{1/2} \varepsilon^j_{k_1,\dots,k_j} \right) + \ln 2$$

$$F_N(\beta) \geqslant \beta N^{-1/2} \operatorname*{Max}_{k_1,\dots,k_n} \left(\sum_{j=1}^{n} a_j^{1/2} \varepsilon^j_{k_1,\dots,k_j} \right)$$

if $\alpha > 1$ we get, calling

$$\xi = \operatorname*{Max}_{k_1,\dots,k_n} \left(\sum_{j=1}^{n} a_j^{1/2} \varepsilon^j_{k_1,\dots,k_j} \right)$$

that

$$\begin{aligned}
&\int_{|F_N|^p > \alpha} |F_N(\beta)|^p \, d\mathbb{P} \\
&\quad \leqslant \int_{\beta\xi/\sqrt{N} + \ln 2 > \alpha^{1/p}} \left(\frac{\beta\xi}{\sqrt{N}} + \ln 2 \right)^p d\mathbb{P} \\
&\quad\quad + \int_{\beta\xi/\sqrt{N} < -\alpha^{1/p}} \left(-\frac{\beta}{\sqrt{N}} \xi \right)^p d\mathbb{P} \\
&\quad \leqslant \sum_{l=1}^{\infty} [(l+1)\alpha^{1/p} + \ln 2]^p \, \mathbb{P}\left(\xi > \frac{l\sqrt{N}(\alpha^{1/p} - \ln 2)}{\beta} \right) \\
&\quad\quad + \sum_{l=1}^{\infty} (l+1)^p \alpha \mathbb{P}\left(\xi \leqslant \frac{-\alpha^{1/2} l \sqrt{N}}{\beta} \right)
\end{aligned}$$

since ξ is the maximum over 2^N random variables, we have that

$$\mathbb{P}\left(\xi > \frac{l\sqrt{N}(\alpha^{1/p} - \ln 2)}{\beta}\right) \leqslant 2^N \mathbb{P}\left(\sum_{j=1}^{n} a_j^{1/2}\varepsilon^j \geqslant \frac{l\sqrt{N}}{\beta}(\alpha^{1/p} - \ln 2)\right)$$
$$\leqslant 2^N \exp -\frac{l^2 N}{2\beta^2}(\alpha^{1/p} - \ln 2)^2$$

if $\alpha^{1/p} - \ln 2 > 1$, where the last inequality follows from the fact that $\sum_{j=1}^{n} a_j^{1/2}\varepsilon^j \in \mathcal{N}(0, 1)$.

Furthermore, it is not difficult to convince oneself that

$$\mathbb{P}\left(\xi < -\frac{\sqrt{N}l\alpha^{1/p}}{\beta}\right) \leqslant 2^N \mathbb{P}\left(\sum_{j=1}^{n} a_j^{1/2}\varepsilon^j \leqslant -\frac{\sqrt{N}l\alpha^{1/p}}{\beta}\right)$$
$$\leqslant 2^N \exp -\frac{Nl^2\alpha^{2/p}}{2\beta^2}$$

Thus, when $\alpha^{1/p} > \ln 2 + 2\beta(\ln 2)^{1/2}$, we get

$$\sup_{N \geqslant 1} \int_{|F_N|^p > \alpha} |F_N|^p \, d\mathbb{P} \leqslant \sum_{l=1}^{\infty} [(l+1)a^{1/p} + \ln 2]^p$$
$$\times \exp -\frac{l^2(\alpha^{1/p} - \ln^2)}{4\beta^2} + \sum_{l=1}^{\infty} (l+1)^p \alpha \exp -\frac{l^2(\alpha^{2/p})}{4\beta^2}$$

which goes to zero when α goes to infinity.

3.2. Explicit Evaluation of the Free Energy

At the end of this section we will give the explicit formula for the free energy for arbitrary sequences $(\ln \alpha_i)_{i=1}^{n}$ and $(a_i)_{i=1}^{n}$ with $\sum_{i=1}^{n} \ln \alpha_i = \ln 2$ and $\sum_{i=1}^{n} a_i = 1$. The problem is just to explicitly evaluate the coordinates of the point m in Theorem 2.1.

Let us first define, if j and k are two integers smaller than n, with $j \leqslant k$,

$$B_{j,k} = 2 \sum_{i=j}^{k} \ln \alpha_i \Big/ \sum_{i=j}^{k} a_i, \qquad B_{j,k} > 0$$

Let us also define the following integers: $J_0^* = 1$,

$$J_1^* = \operatorname{Inf}\{J > 1 \mid B_{1,J}^2 < B_{J+1,l}^2, \ \forall l \geqslant J+1\}$$
$$\vdots$$
$$J_k^* = \operatorname{Inf}\{J > J_{k-1}^* \mid B_{J_{k-1}^*,J}^2 < B_{J+1,l}^2, \ \forall l \geqslant J+1\}$$

and let $\beta_k > 0$ be such that

$$\beta_k^2 = B^2_{J^*_{k-1}, J^*_k}$$

From the very definition of J^*_k it follows that β_k is an increasing sequence of real numbers. For $\beta_k \leqslant \beta \leqslant \beta_{k+1}$, we will prove that the point $\tilde{m}$ in $\mathbb{R}^n$ with coordinates

$$\begin{aligned} \tilde{m}_i &= \beta_l a_i^{1/2} \qquad \text{if} \quad i \in [J^*_{l-1}+1, ..., J^*_l], \quad l \in 1, ..., k \\ \tilde{m}_i &= \beta\sqrt{a_i} \qquad \text{if} \quad i \in [J^*_k + 1, ..., n] \end{aligned}$$

is the point m of the compact subset $\tilde{\mathscr{E}}(m)$ at minimal distance from $m^* = (\beta a_i^{1/2})_{i=1}^n$.

Starting from the vectorial identity in $\mathbb{R}^n$

$$(\varepsilon - m^*)^2 - (\varepsilon - \tilde{m})^2 - (\tilde{m} - m^*)^2 = -2(\varepsilon - \tilde{m}) \cdot (m^* - \tilde{m})$$

if we can prove that for any $\varepsilon \in \tilde{\mathscr{E}}(n)$

$$\mathscr{L}(\varepsilon) = (\varepsilon - \tilde{m}) \cdot (m^* - \tilde{m}) \leqslant 0$$

then $\tilde{m} = m$ if $\beta \in [\beta_k, \beta_{k+1}]$.

Since $\tilde{m}_i = m_i^*$, $\forall i \geqslant J^*_k + 1$,

$$\begin{aligned} \mathscr{L}(\varepsilon) &= \sum_{i=1}^{J^*_k} (\varepsilon_i - \tilde{m}_i)(m_i^* - \tilde{m}_i) \\ &= \sum_{l=1}^{k} \left(\frac{\beta}{\beta_l} - 1\right) \sum_{i=J^*_{l-1}}^{J^*_l} (\varepsilon_i - \tilde{m}_i) m_i \end{aligned}$$

Let us remark that if

$$\varepsilon \in \tilde{\mathscr{E}}(J^*_k) = \bigcap_{j=1}^{J^*_k} S(j)$$

where

$$S(j) = \left\{ \varepsilon \Big/ \sum_{i=1}^{j} \varepsilon_i^2 \leqslant \sum_{i=1}^{j} \ln \alpha_i \right\}$$

then

$$\varepsilon \in \bigcap_{j=1}^{J^*_k} \left\{ \varepsilon / g_j(\varepsilon) \equiv \sum_{i=1}^{j} (\varepsilon_i - \tilde{m}_i)\tilde{m}_i \leqslant 0 \right\}$$

since $g_j(\varepsilon)=0$ is the equation of the tangent hyperplane at $\tilde{m}$ to $S(j)$. Using

$$\sum_{J^*_{l-1}}^{J^*_l} (\varepsilon_i-\tilde{m}_i)\tilde{m}_i = g_{J^*_l} - g_{J^*_{l-1}}$$

with

$$g_{J^*_0}=0$$

we get

$$\begin{aligned}
\mathscr{L}(\varepsilon) &= \sum_{l=1}^{k}\left(\frac{\beta}{\beta_l}-1\right)(g_{J^*_l}-g_{J^*_{l-1}})\\
&= -g_{J^*_k}+\sum_{l=1}^{k}\frac{\beta}{\beta_l}(g_{J^*_l}-g_{J^*_{l-1}})\\
&= \left(\frac{\beta}{\beta_k}-1\right)g_{J^*_k}+\sum_{l=1}^{k-1}\beta\left(\frac{1}{\beta_l}-\frac{1}{\beta_{l+1}}\right)g_{J^*_l}\leqslant 0
\end{aligned}$$

where we have used $\beta_l<\beta_{l+1}$ and $g_J(\varepsilon)\leqslant 0$, $\forall J\in 1,..., J^*_k$ if $\varepsilon\in\tilde{\mathscr{E}}(J^*_k)$. Thus

$$F_n(\beta)=\frac{1}{2}\|m^*\|^2-\frac{1}{2}\|m-m^*\|^2+\ln 2$$

$$=\begin{cases}
\frac{1}{2}\beta^2+\ln 2 \qquad \text{if}\quad \beta\leqslant\beta_1\\
\sum_{l=1}^{k}\beta_l\beta\left(\sum_{i=J^*_{l-1}+1}^{J^*_l} a_i\right)+\sum_{i=J^*_k+1}^{n}\left(\frac{1}{2}\beta^2 a_i+\ln\alpha_i\right)\\
\quad \text{if}\quad \beta_k\leqslant\beta\leqslant\beta_{k+1}\quad \forall k\in 1,..., l(n)\\
\sum_{l=1}^{l(n)}\beta_l\left(\sum_{i=J^*_{l-1}+1}^{J^*_l} a_i\right) \qquad \text{if}\quad \beta\geqslant\beta_l(n)
\end{cases}$$

ACKNOWLEDGMENTS

D.C. wishes to thank the CNRS and the CPT Marseille, Luminy, for their very kind hospitality. M.C. thanks the Université de Provence for financial support and CPT, Marseille, Luminy for their very kind hospitality. P.P. thanks the INFN, Sezione di Roma, for financial support and the Dipartimento di Fisica G. Marconi for their very kind hospitality. We thanks S. Miracle Sole and J. Ruiz for useful conversations.

REFERENCES

1. B. Derrida, *Phys. Rev . Lett.* **45**:79–82 (1980); *Phys. Rev. B* **24**:2613–2626 (1981); *J. Phys. Lett.* (Paris) **46**:L401 (1985).
2. B. Derrida and E. Gardner, Preprint Saclay (1985).
3. M. Mezard and D. Gross, *Nucl. Phys. B* **240**:431 (1984).
4. M. Sherrington and S. Kirkpatrick, *Phys. Rev. Lett.* **32**:1792 (1975).
5. E. Olivieri and P. Picco, *Commun. Math. Phys.* **96**:125–144 (1984).
6. Y. S. Chow and M. Teicher, *Probability Theory* (Springer, Berlin, 1978).